Motivation

Many tasks in NLP can be cast as sequence labeling, where each token (usually, word) gets its own label. Compare:

- **Text classification:** \( \langle x_1, x_2, \ldots, x_n \rangle \mapsto y \in L \)
- **Sequence labeling:** \( \langle x_1 \mapsto y_1, x_2 \mapsto y_2, \ldots, x_n \mapsto y_n \rangle \), each \( y_i \in L \)
- **Translation:** \( x \mapsto y \in V_{\text{target}}^* \)
Problems Typically Cast as Sequence Labeling

- supersense tagging (Ciaramita and Johnson, 2003)
- part-of-speech tagging (Church, 1988)
- morphosyntactic tagging (Habash and Rambow, 2005)
- segmentation into words (Sproat et al., 1996) or multiword expressions (Schneider et al., 2014)
- code switching (Solorio and Liu, 2008)
- dialogue acts (Stolcke et al., 2000)
- spelling correction (Kernighan et al., 1990)
- word alignment (Vogel et al., 1996)
- named entity recognition (Bikel et al., 1999)
- compression (Conroy and O'Leary, 2001)
Example Problem: Supersenses

A problem with a long history: word-sense disambiguation.
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- E.g., from a dictionary
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Ciaramita and Johnson (2003) and Ciaramita and Altun (2006) used a lexicon called WordNet to define 41 semantic classes for words.

▶ WordNet (Fellbaum, 1998) is a fascinating resource in its own right! See http://wordnetweb.princeton.edu/perl/webwn to get an idea.
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This represents a coarsening of the annotations in the Semcor corpus (Miller et al., 1993).
Example: *box’s Thirteen Synonym Sets, Eight Supersenses*

1. box: a (usually rectangular) container; may have a lid. “he rummaged through a box of spare parts”
2. box/loge: private area in a theater or grandstand where a small group can watch the performance. “the royal box was empty”
3. box/boxful: the quantity contained in a box. “he gave her a box of chocolates”
4. corner/box: a predicament from which a skillful or graceful escape is impossible. “his lying got him into a tight corner”
5. box: a rectangular drawing. “the flowchart contained many boxes”
6. box/boxwood: evergreen shrubs or small trees
7. box: any one of several designated areas on a ball field where the batter or catcher or coaches are positioned. “the umpire warned the batter to stay in the batter’s box”
8. box/box seat: the driver’s seat on a coach. “an armed guard sat in the box with the driver”
9. box: separate partitioned area in a public place for a few people. “the sentry stayed in his box to avoid the cold”
10. box: a blow with the hand (usually on the ear). “I gave him a good box on the ear”
11. box/package: put into a box. “box the gift, please”
12. box: hit with the fist. “I’ll box your ears!”
Example: box’s Thirteen Synonym Sets, Eight Supersenses

1. box: a (usually rectangular) container; may have a lid. “he rummaged through a box of spare parts” ⇝ N.ARTIFACT
2. box/loge: private area in a theater or grandstand where a small group can watch the performance. “the royal box was empty” ⇝ N.ARTIFACT
3. box/boxful: the quantity contained in a box. “he gave her a box of chocolates” ⇝ N.QUANTITY
4. corner/box: a predicament from which a skillful or graceful escape is impossible. “his lying got him into a tight corner” ⇝ N.STATE
5. box: a rectangular drawing. “the flowchart contained many boxes” ⇝ N.SHAPE
6. box/boxwood: evergreen shrubs or small trees ⇝ N.PLANT
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11. box/package: put into a box. “box the gift, please” ⇝ V.CONTACT
12. box: hit with the fist. “I’ll box your ears!” ⇝ V.CONTACT
13. box: engage in a boxing match. ⇝ V.COMpetition
Clara Harris, one of the guests in the box, stood up and demanded water.
Observations

▶ Lots of subproblems: Which words have supersenses? Which words group together to form a multiword expression? For those that do, which supersense?
▶ Every word’s label depends on the words around it, and their labels.
▶ Segmentation problems can be cast as sequence labeling (Ramshaw and Marcus, 1995):
  ▶ Two labels, B and I, if every word must be in some segment
  ▶ Three labels, B, I, and O, if some words are to be “discarded”
  ▶ Variants for five labels (E for end, S for singleton), gaps/noncontiguous spans, and nesting, exist.
Concatenate B, I, etc., with labels to get labeled segmentation.
▶ Some sequences of labels might be invalid under your theory/label semantics.
▶ Evaluation: usually precision, recall, and $F_1$ on labeled segments.
Every linguistic analyzer is comprised of:

1. Theoretical motivation from linguistics and/or the text domain
2. An algorithm that maps $\mathcal{V}^\dagger$ to some output space $\mathcal{Y}$.
   ▶ Some $\mathcal{Y}$ are very specialized, but others, like the one we discuss here, show up again and again.
3. An implementation of the algorithm
   ▶ Once upon a time: rule systems and crafted rules
   ▶ More robust: supervised learning from annotated data
   ▶ Today: unsupervised pretraining followed by supervised finetuning
Problem statement: given a sequence of \( n \) words \( x \), assign each a label from \( \mathcal{L} \). Let \( L = |\mathcal{L}| \).

Every approach we see today will cast the problem as:

\[
\hat{y} = \arg\max_{y \in \mathcal{L}^n} \text{Score}(x, y; \theta)
\]

Naïvely, that’s a classification problem where the number of possible ‘labels’ (output sequences) depends on the input and is \( O(L^n) \) in size!
Define score of a word \( x_i \) getting label \( y \in \mathcal{L} \) in context:
\[
\text{score}(x, i, y; \theta),
\]
for example through a feature vector, \( f(x, i, y) \).
(Here, “\( i \)” indicates the position of the input word to be classified.)

Train a classifier to decode locally, i.e.,

\[
\hat{y}_i = \operatorname{argmax}_{y \in \mathcal{L}} \text{score}(x, i, y; \theta)
\]

\[
\overset{\text{MLR}}{=} \operatorname{argmax}_{y \in \mathcal{L}} \theta^\top f(x, i, y)
\]

The classifier is applied to each \( x_1, x_2, \ldots \) in turn, but all the words can be made available at each position.
Sequence Labeling v. 0: Local Classifiers

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We can do better when there are predictable relationships among labels.
Reflection

If we return to the original formulation,

$$\hat{y} = \arg\max_{y \in \mathcal{L}^n} \text{Score}(x, y; \theta),$$

how can we write “Score” in terms of the notation on the last slide?
Local Classifiers (v. 0)

Lightweight; no need to learn anything new! But labels can’t affect each other.
Sequence Labeling v. 1: Sequential Classifiers

Define score of a word $x_i$ getting label $y$ in context, including previous labels: $\text{score}(x, i, \hat{y}_{1:i-1}, y; \theta)$. (From here, we won’t always write $\theta$, but the dependence remains.)

Train a classifier, e.g.,

$$\hat{y}_i = \arg\max_{y \in \mathcal{L}} \text{score}(x, i, \hat{y}_{1:i-1}, y)$$

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There is much literature on methods for training, and for decoding, with models like this. Important decoding method in NLP: beam search.
Beam Search for Sequential Classifiers

Input: $x$ (length $n$), a sequential classifier’s scoring function $\text{score}$, and beam width $k$

Let $H_0$ score hypotheses at position $0$, defining only $H_0(\langle \rangle) = 0$. For $i \in \{1, \ldots, n\}$:

- Empty $C$.
- For each hypothesis $\hat{y}_{1:i-1}$ scored by $H_{i-1}$:
  - For each $y \in \mathcal{L}$, place new hypothesis $\hat{y}_{1:i-1} y \rightarrow H_{i-1}(\hat{y}_{1:i-1}) + \text{score}(x, i, \hat{y}_{1:i-1}, y)$ into $C$.
- Let $H_i$ be the $k$-best scored elements of $C$.

Output: best scored element of $H_n$. 

Notes on Beam Search for Sequential Classifiers

- Runtime is $O(n^2kL)$, space is $O(n^2k)$.
- You can improve runtime (e.g., to $O(nkL)$) if computation is shared across different $i$ (often true with neural networks).
- Special cases:
  - $k = 1$ is greedy left-to-right decoding.
  - At $k = L^n$, you're doing brute force, exhaustive search.
- Generally: no guarantee.
Suppose your label set is built out of BIO tags. For an output $\hat{y}$ to be well-formed, it suffices to ensure that it contains no “Ol” label bigrams.

How would you modify beam search to guarantee well-formedness?
Very powerful! Algorithms lack guarantees.
A Generative Approach

The next approach should remind you of language models. It assumes that labeled sequences are generated according to the following story:
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\uparrow
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\[
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\]
\[
\uparrow \quad \uparrow
\]
\[
y_1 \quad y_2
\]

\[
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x_1 \quad x_2 \quad x_3 \\
\uparrow \quad \uparrow \quad \uparrow \\
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\[
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\[ \uparrow \quad \uparrow \quad \uparrow \]
\[ y_1 \rightarrow y_2 \rightarrow y_3 \rightarrow y_4 \]

\[ y_4 \sim p_{transition}(Y \mid y_3) \]
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x_1 \quad x_2 \quad x_3 \quad x_4 \\
\uparrow \quad \uparrow \quad \uparrow \quad \uparrow \\
y_1 \quad \rightarrow \quad y_2 \quad \rightarrow \quad y_3 \quad \rightarrow \quad y_4
\]

\[x_4 \sim p_{\text{emission}}(X \mid y_4)\]
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\[
\begin{align*}
\uparrow & \uparrow & \uparrow & \uparrow \\
y_1 & \rightarrow & y_2 & \rightarrow & y_3 & \rightarrow & y_4 & \rightarrow & 8 \\
x_1 & x_2 & x_3 & x_4 \\
y_5 & \sim & P_{transition}(Y \mid y_4)
\end{align*}
\]
By convention, $y_{n+1} = \bigcirc$ is always the “stop label.”

\[
p(X = x, Y = y) = p_{\text{start}}(y_1) \cdot 
\prod_{i=1}^{n} p_{\text{emission}}(x_i \mid y_i) \cdot p_{\text{transition}}(y_{i+1} \mid y_i)
\]

\[
\hat{y} = \arg\max_{y \in \mathcal{L}^n} p(Y = y \mid X = x)
\]

\[
= \arg\max_{y \in \mathcal{L}^n} p(X = x, Y = y)
\]

\[
= \arg\max_{y \in \mathcal{L}^n} \log p(X = x, Y = y)
\]

We can solve the global decoding problem exactly (i.e., find the model-optimal $\hat{y}$) in $O(nL^2)$ time and $O(nL)$ space using the Viterbi algorithm (more later).
HMM Parameters

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$p_{\text{emission}}$ is a distribution over words, for each label. Many people find this counterintuitive! Estimation: counting occurrences of labels with words, and normalizing (per label, not per word).

$p_{\text{transition}}$ is exactly a bigram (first-order Markov) model over labels.
Classical HMMs vs. Classifiers

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- For humans: choosing features or designing a neural architecture that can learn good features
Classical HMMs vs. Classifiers

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▶ For machines: estimating the parameters (typically by SGD); (in the sequential case) searching for “argmax”
Classical HMMs vs. Classifiers

With classifiers (local or sequential), the hard work is:

- For humans: choosing features or designing a neural architecture that can learn good features
- For machines: estimating the parameters (typically by SGD); (in the sequential case) searching for “argmax”

With classical HMMs, the parameters \((p_{\text{transition}}, p_{\text{emission}}, p_{\text{start}})\) have a closed form if you have labeled data! The hardest part is implementing the algorithm for choosing the “argmax” label sequence. Downside:

- You don’t get to design or learn features.
Reflection

The runtime of the model-optimal decoding algorithm for HMMs depends quadratically on the size of \( \mathcal{L} \). For some problems (e.g., supersense tagging) the label set can be large. Can you think of a way to trade the guarantee of model-optimality for speed, while still using the HMM?
Hidden Markov Models (v. 2)

Algorithmically beautiful; lack of features is unsatisfying.
To endow HMMs with features, we can replace the “lookup” probabilities \( (p_{transition}, p_{emission}, p_{start}) \) with scoring functions. This idea was explored by Berg-Kirkpatrick et al. (2010).

Classical HMM (v. 2):

\[
\hat{y} = \arg\max_{y \in \mathcal{L}^n} \log p_{start}(y_1) + \sum_{i=1}^{n} \left( \log p_{emission}(x_i | y_i) + \log p_{transition}(y_{i+1} | y_i) \right)
\]

This approach (v. 3):

\[
\hat{y} = \arg\max_{y \in \mathcal{L}^n} s_{start}(y_1) + \sum_{i=1}^{n} s_{emission}(x_i, y_i) + s_{transition}(y_i, y_{i+1})
\]

Each “s” could be a linear scoring function (like in MLR), perhaps using word or label vectors. For now, I’m hiding the parameters of each s.
Decoding is essentially the same as the HMM: Viterbi algorithm.
Notes on V. 3

- Decoding is essentially the same as the HMM: Viterbi algorithm.
- Learning is now complicated and depends on the form of each “s,” though I promise each iteration will be efficient. (Put this on my tab, along with Viterbi.)
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No part of the the scoring function looks at neighboring words.
Brings features to HMMs, but learning is going to require more than just counting and normalizing.
Sequence Labeling v. 4

Let each scoring component ("s") “see” the whole input. By convention, $y_0 = \bigcirc$ is always the “start label.”

$$\hat{y} = \arg\max_{y \in \mathcal{L}^n} \sum_{i=0}^{n} s(x, i, y_i, y_{i+1})$$

Note that $x$ can have arbitrary length, so we need “s” functions that are capable of adapting to variable-length input.
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Notes on V. 4

- Decoding is essentially the same as the HMM and v. 3: Viterbi algorithm.
- As with v. 3, learning is complicated and depends on the form of each “s.”
- This model strictly generalizes local classifiers (v. 0), the HMM (v. 2), and v. 3.
Even better features for HMMs, with the promise of efficient decoding and learning.
Claim: As we move from v. 1 (sequential classifiers) to v. 4 to v. 0 (local classifiers), the scoring functions available become strictly less expressive.
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Compare v. 1 and v. 4. What kinds of features can you use in v. 1 that you can’t use in v. 4?
Reflection

Claim: As we move from v. 1 (sequential classifiers) to v. 4 to v. 0 (local classifiers), the scoring functions available become strictly less expressive.

Compare v. 1 and v. 4. What kinds of features can you use in v. 1 that you can’t use in v. 4?

Now consider v. 4 and v. 0. What kinds of features can you use in v. 4 that you can’t use in v. 0?
Where We Are

<table>
<thead>
<tr>
<th>Score decomp.</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( s(x, i, y_i) )</td>
<td>( s(x, i, y_{1:i}) )</td>
<td>emission/transition</td>
<td>( s(x_i, y_i) + s(y_i, y_{i+1}) )</td>
<td>( s(x, i, y_i, y_{i+1}) )</td>
<td></td>
</tr>
<tr>
<td>learn</td>
<td>SGD</td>
<td>?</td>
<td>count &amp; normalize</td>
<td>?</td>
<td>?</td>
</tr>
<tr>
<td>decode</td>
<td>local</td>
<td>beam search</td>
<td>Viterbi</td>
<td>Viterbi</td>
<td>Viterbi</td>
</tr>
</tbody>
</table>
The Main Dish
Two Problems to Solve

1. Decoding: the Viterbi algorithm for choosing $\hat{y}$.
   - Usually taught for classical HMMs (v. 2); I will teach it for v. 4, abstracting away “$s$.”

2. Learning: estimating the parameters of each $s$ function.
   - Depending on your choices here, you arrive at the structured perceptron, the classical conditional random field (CRF), neural CRFs, and more.
A Data Structure

labels in $\mathcal{L}$

<table>
<thead>
<tr>
<th></th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$\ldots$</th>
<th>$x_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\ell_1$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\ell_2$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\vdots$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\ell_L$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The cell at row $j$, column $i$ will hold information pertaining to choosing $\hat{y}_i = \ell_j$. 
The End of the Sequence

\[
\begin{array}{cccc}
\text{input sequence} & x_1 & x_2 & \ldots & x_n \\
\hline
\ell_1 & & & & \\
\ell_2 & & & & \\
\vdots & & & & \\
\ell_L & & & & \\
\end{array}
\]

labels in \( \mathcal{L} \)

\[
\hat{y}_n = \arg\max_{y_n \in \mathcal{L}} \sum_{i=0}^{n} s(x, i, y_i, y_{i+1})
\]

\[
= \arg\max_{y_n \in \mathcal{L}} s(x, i, y_{n-1}, y_n) + s(x, i, y_n, \bigcirc)
\]

The decision about \( \hat{y}_n \) is a function of \( y_{n-1}, x \), and nothing else!
High-Level View of the Viterbi Algorithm

- The decision about $\hat{y}_n$ is a function of $y_{n-1}$, $x$, and nothing else!
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If, for each value of $y_{n-1}$, we knew the best $(n - 1)$-length label prefix $y_{1:n-1}$, then picking $\hat{y}_n$ (and $\hat{y}_{n-1}$) would be easy.
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Idea: for each position $i$, calculate the score of the best label prefix $y_{1:i}$ ending in each possible value for the $i$th label.

We’ll call this value $\heartsuit_i(\ell)$ for $y_i = \ell$. 
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With a little bookkeeping, we can then trace backwards and recover the best label sequence.
Recurrence

First, think about the score of the best sequence.

Let \( \heartsuit_{i}(y) \) be the score of the best label sequence for \( x_{1:i} \) that ends in \( y \). It is defined recursively:

\[
\heartsuit_{n+1}(\bigcirc) = \max_{y_n \in \mathcal{L}} s(x, n, y_n, \bigcirc) + \heartsuit_n(y_n)
\]
Recurrence

First, think about the score of the best sequence.

Let $\circ_n(y)$ be the score of the best label sequence for $x_{1:i}$ that ends in $y$. It is defined recursively:

\[
\circ_{n+1}(\bigcirc) = \max_{y_n \in \mathcal{L}} s(x, n, y_n, \bigcirc) + \circ_n(y_n)
\]

\[
\circ_n(y) = \max_{y_{n-1} \in \mathcal{L}} s(x, n - 1, y_{n-1}, y) + \circ_{n-1}(y_{n-1})
\]
Recurrence

First, think about the score of the best sequence.

Let $\diamondsuit_i(y)$ be the score of the best label sequence for $x_{1:i}$ that ends in $y$. It is defined recursively:

$$
\diamondsuit_{n+1} = \max_{y_n \in \mathcal{L}} s(x, n, y_n, \Diamond) + \diamondsuit_n(y_n)
$$

$$
\diamondsuit_n(y) = \max_{y_{n-1} \in \mathcal{L}} s(x, n - 1, y_{n-1}, y) + \diamondsuit_{n-1}(y_{n-1})
$$

$$
\diamondsuit_{n-1}(y) = \max_{y_{n-2} \in \mathcal{L}} s(x, n - 2, y_{n-2}, y) + \diamondsuit_{n-2}(y_{n-2})
$$
Recurrence

First, think about the score of the best sequence.

Let $\diamondsuit_i(y)$ be the score of the best label sequence for $x_{1:i}$ that ends in $y$. It is defined recursively:

\[
\diamondsuit_{n+1}(\emptyset) = \max_{y_n \in \mathcal{L}} s(x, n, y_n, \emptyset) + \diamondsuit_n(y_n)
\]

\[
\diamondsuit_n(y) = \max_{y_{n-1} \in \mathcal{L}} s(x, n - 1, y_{n-1}, y) + \diamondsuit_{n-1}(y_{n-1})
\]

\[
\diamondsuit_{n-1}(y) = \max_{y_{n-2} \in \mathcal{L}} s(x, n - 2, y_{n-2}, y) + \diamondsuit_{n-2}(y_{n-2})
\]

\[\vdots\]

\[
\diamondsuit_i(y) = \max_{y_{i-1} \in \mathcal{L}} s(x, i - 1, y_{i-1}, y) + \diamondsuit_{i-1}(y_{i-1})
\]
Recurrence

First, think about the score of the best sequence.

Let $\diamondsuit_i(y)$ be the score of the best label sequence for $x_{1:i}$ that ends in $y$. It is defined recursively:

\[
\begin{align*}
\diamondsuit_{n+1}(\bigcirc) &= \max_{y_n \in \mathcal{L}} s(x, n, y_n, \bigcirc) + \diamondsuit_{n}(y_n) \\
\diamondsuit_n(y) &= \max_{y_{n-1} \in \mathcal{L}} s(x, n - 1, y_{n-1}, y) + \diamondsuit_{n-1}(y_{n-1}) \\
\diamondsuit_{n-1}(y) &= \max_{y_{n-2} \in \mathcal{L}} s(x, n - 2, y_{n-2}, y) + \diamondsuit_{n-2}(y_{n-2}) \\
& \vdots \\
\diamondsuit_i(y) &= \max_{y_{i-1} \in \mathcal{L}} s(x, i - 1, y_{i-1}, y) + \diamondsuit_{i-1}(y_{i-1}) \\
& \vdots \\
\diamondsuit_1(y) &= s(x, 0, \bigcirc, y)
\end{align*}
\]
Viterbi Procedure (Part I: Prefix Scores)

<table>
<thead>
<tr>
<th></th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$\ldots$</th>
<th>$x_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\ell_1$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\ell_2$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\vdots$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\ell_L$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Input sequence $x_1, x_2, \ldots, x_n$
Viterbi Procedure (Part I: Prefix Scores)

<table>
<thead>
<tr>
<th>( L )</th>
<th>( \ell_1 )</th>
<th>( \ell_2 )</th>
<th>( \ldots )</th>
<th>( \ell_L )</th>
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</thead>
<tbody>
<tr>
<td>( l_1 )</td>
<td>( \heartsuit_1(l_1) )</td>
<td>( \heartsuit_1(l_2) )</td>
<td>( \heartsuit_1(l_L) )</td>
<td></td>
</tr>
<tr>
<td>( \ell_2 )</td>
<td>( \heartsuit_1(l_1) )</td>
<td>( \heartsuit_1(l_2) )</td>
<td>( \heartsuit_1(l_L) )</td>
<td></td>
</tr>
<tr>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td></td>
</tr>
<tr>
<td>( \ell_L )</td>
<td>( \heartsuit_1(l_1) )</td>
<td>( \heartsuit_1(l_2) )</td>
<td>( \heartsuit_1(l_L) )</td>
<td></td>
</tr>
<tr>
<td>( x_1 )</td>
<td>( x_2 )</td>
<td>( \ldots )</td>
<td>( x_n )</td>
<td></td>
</tr>
</tbody>
</table>

\[ \heartsuit_1(y) = s(\mathbf{x}, 0, \bigcirc, y) \]
Viterbi Procedure (Part I: Prefix Scores)

<table>
<thead>
<tr>
<th>$\ell_1$</th>
<th>$\heartsuit_1(\ell_1)$</th>
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<th></th>
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<tbody>
<tr>
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<td></td>
<td></td>
</tr>
<tr>
<td>$\vdots$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\ell_L$</td>
<td>$\heartsuit_1(\ell_L)$</td>
<td>$\heartsuit_2(\ell_L)$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[ \heartsuit_i(y) = \max_{y_{i-1} \in \mathcal{L}} s(x, i - 1, y_{i-1}, y) + \heartsuit_{i-1}(y_{i-1}) \]
Viterbi Procedure (Part I: Prefix Scores)

<table>
<thead>
<tr>
<th>$\ell_1$</th>
<th>$\bigtriangleup_1(\ell_1)$</th>
<th>$\bigtriangleup_2(\ell_1)$</th>
<th>$\ldots$</th>
<th>$\bigtriangleup_n(\ell_1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\ell_2$</td>
<td>$\bigtriangleup_1(\ell_2)$</td>
<td>$\bigtriangleup_2(\ell_2)$</td>
<td>$\ldots$</td>
<td>$\bigtriangleup_n(\ell_2)$</td>
</tr>
<tr>
<td>$\vdots$</td>
<td>$\ldots$</td>
<td>$\ldots$</td>
<td>$\ldots$</td>
<td>$\ldots$</td>
</tr>
<tr>
<td>$\ell_L$</td>
<td>$\bigtriangleup_1(\ell_L)$</td>
<td>$\bigtriangleup_2(\ell_L)$</td>
<td>$\ldots$</td>
<td>$\bigtriangleup_n(\ell_L)$</td>
</tr>
</tbody>
</table>

$$\bigtriangleup_n(y) = \max_{y_{n-1} \in \mathcal{L}} s(x, n-1, y_{n-1}, y) + \bigtriangleup_{n-1}(y_{n-1})$$
Viterbi Procedure (Part I: Prefix Scores)

<table>
<thead>
<tr>
<th>( \ell_1 )</th>
<th>( x_1 )</th>
<th>( \heartsuit_1(\ell_1) )</th>
<th>( x_2 )</th>
<th>( \heartsuit_2(\ell_1) )</th>
<th>( \ldots )</th>
<th>( x_n )</th>
<th>( \heartsuit_n(\ell_1) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \ell_2 )</td>
<td>( \heartsuit_1(\ell_2) )</td>
<td>( \heartsuit_2(\ell_2) )</td>
<td>( \ldots )</td>
<td>( \heartsuit_n(\ell_2) )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \ell_L )</td>
<td>( \heartsuit_1(\ell_L) )</td>
<td>( \heartsuit_2(\ell_L) )</td>
<td>( \ldots )</td>
<td>( \heartsuit_n(\ell_L) )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \bigcirc )</td>
<td>( \bigcirc )</td>
<td>( \bigcirc )</td>
<td>( \bigcirc )</td>
<td>( \bigcirc )</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\( \heartsuit_{n+1}(\bigcirc) = \max_{y_n \in \mathcal{L}} s(\mathbf{x}, n, y_n, \bigcirc) + \heartsuit_n(y_n) \)
High-Level View of the Viterbi Algorithm

- The decision about $\hat{y}_n$ is a function of $y_{n-1}$, $x$, and nothing else!
- If, for each value of $y_{n-1}$, we knew the best $(n-1)$-length label prefix $y_{1:n-1}$, then picking $\hat{y}_n$ (and $\hat{y}_{n-1}$) would be easy.
- Idea: for each position $i$, calculate the score of the best label prefix $y_{1:i}$ ending in each possible value for the $i$th label.
  - We’ll call this value $\diamondsuit_i(\ell)$ for $y_i = \ell$.
- With a little bookkeeping, we can then trace backwards and recover the best label sequence.
Viterbi Procedure (Part I: Prefix Scores and Backpointers)

<table>
<thead>
<tr>
<th>(L)</th>
<th>(x_1)</th>
<th>(x_2)</th>
<th>(\ldots)</th>
<th>(x_n)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(l_1)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(l_2)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\vdots)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(l_L)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The table represents the input sequence \(x = (x_1, x_2, \ldots, x_n)\) and the prefix scores \(\ell_1, \ell_2, \ldots, \ell_L\). The diagram shows the backpointers used in the Viterbi algorithm.
Viterbi Procedure (Part I: Prefix Scores and Backpointers)

<table>
<thead>
<tr>
<th></th>
<th>input sequence</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td>$\ell_1$</td>
<td>$\hat{1}(\ell_1)$</td>
<td>bp$_1(\ell_1)$</td>
<td></td>
</tr>
<tr>
<td>$x_2$</td>
<td>$\ell_2$</td>
<td>$\hat{1}(\ell_2)$</td>
<td>bp$_1(\ell_2)$</td>
<td></td>
</tr>
<tr>
<td>$\vdots$</td>
<td>$\ell_L$</td>
<td>$\hat{1}(\ell_L)$</td>
<td>bp$_1(\ell_L)$</td>
<td></td>
</tr>
<tr>
<td>$\varnothing$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

$\hat{1}(y) = s(\varnothing, 0, \varnothing, y)$

bp$_1(y) = \varnothing$
Viterbi Procedure (Part I: Prefix Scores and Backpointers)

<table>
<thead>
<tr>
<th></th>
<th>(x_1)</th>
<th>(x_2)</th>
<th>(\ldots)</th>
<th>(x_n)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\ell_1)</td>
<td>(\bigvee_1(\ell_1))</td>
<td>(\bigvee_2(\ell_1))</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(bp_1(\ell_1))</td>
<td>(bp_2(\ell_1))</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\ell_2)</td>
<td>(\bigvee_1(\ell_2))</td>
<td>(\bigvee_2(\ell_2))</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(bp_1(\ell_2))</td>
<td>(bp_2(\ell_2))</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\ell_L)</td>
<td>(\bigvee_1(\ell_L))</td>
<td>(\bigvee_2(\ell_L))</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(bp_1(\ell_L))</td>
<td>(bp_2(\ell_L))</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[
\bigvee_i(y) = \max_{y_{i-1} \in \mathcal{L}} s(\mathbf{x}, i - 1, y_{i-1}, y) + \bigvee_{i-1}(y_{i-1})
\]

\[
bp_i(y) = \text{argmax}_{y_{i-1} \in \mathcal{L}} s(\mathbf{x}, i - 1, y_{i-1}, y) + \bigvee_{i-1}(y_{i-1})
\]
Viterbi Procedure (Part I: Prefix Scores and Backpointers)

input sequence

<table>
<thead>
<tr>
<th></th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>...</th>
<th>$x_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$l_1$</td>
<td>$\bigtriangledown_1(l_1)$</td>
<td>$\bigtriangledown_2(l_1)$</td>
<td>...</td>
<td>$\bigtriangledown_n(l_1)$</td>
</tr>
<tr>
<td></td>
<td>$bp_1(l_1)$</td>
<td>$bp_2(l_1)$</td>
<td>...</td>
<td>$bp_n(l_1)$</td>
</tr>
<tr>
<td>$l_2$</td>
<td>$\bigtriangledown_1(l_2)$</td>
<td>$\bigtriangledown_2(l_2)$</td>
<td>...</td>
<td>$\bigtriangledown_n(l_2)$</td>
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<tr>
<td></td>
<td>$bp_1(l_2)$</td>
<td>$bp_2(l_2)$</td>
<td>...</td>
<td>$bp_n(l_2)$</td>
</tr>
<tr>
<td>$\mathcal{L}$</td>
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<td>$\vdots$</td>
<td>...</td>
<td>$\vdots$</td>
</tr>
<tr>
<td>$l_L$</td>
<td>$\bigtriangledown_1(l_L)$</td>
<td>$\bigtriangledown_2(l_L)$</td>
<td>...</td>
<td>$\bigtriangledown_n(l_L)$</td>
</tr>
<tr>
<td></td>
<td>$bp_1(l_L)$</td>
<td>$bp_2(l_L)$</td>
<td>...</td>
<td>$bp_n(l_L)$</td>
</tr>
</tbody>
</table>

$\bigtriangledown_n(y) = \max_{y_{n-1} \in \mathcal{L}} s(x, n - 1, y_{n-1}, y) + \bigtriangledown_{n-1}(y_{n-1})$

$bp_n(y) = \arg\max_{y_{n-1} \in \mathcal{L}} s(x, n - 1, y_{n-1}, y) + \bigtriangledown_{n-1}(y_{n-1})$
Viterbi Procedure (Part I: Prefix Scores and Backpointers)

<table>
<thead>
<tr>
<th></th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>...</th>
<th>$x_n$</th>
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<tbody>
<tr>
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<td>$\mathcal{O}_2(l_1)$</td>
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</tr>
<tr>
<td>bp</td>
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<td>$bp_n(l_1)$</td>
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</tr>
<tr>
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<td>$\mathcal{O}_2(l_2)$</td>
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<tr>
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<tr>
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</tr>
<tr>
<td>$l_L$</td>
<td>$\mathcal{O}_1(l_L)$</td>
<td>$\mathcal{O}_2(l_L)$</td>
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<tr>
<td>bp</td>
<td>$bp_1(l_L)$</td>
<td>$bp_2(l_L)$</td>
<td>$bp_n(l_L)$</td>
<td></td>
</tr>
<tr>
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<td></td>
<td>$\mathcal{O}_{n+1}(\bigcirc)$</td>
</tr>
<tr>
<td>bp</td>
<td></td>
<td></td>
<td></td>
<td>$bp_{n+1}(\bigcirc)$</td>
</tr>
</tbody>
</table>

$$\mathcal{O}_{n+1} (\bigcirc) = \max_{y_n \in \mathcal{L}} s(\mathbf{x}, n, y_n, \bigcirc) + \mathcal{O}_{n}(y_n)$$

$$bp_{n+1}(\bigcirc) = \arg\max_{y_n \in \mathcal{L}} s(\mathbf{x}, n, y_n, \bigcirc) + \mathcal{O}_{n}(y_n)$$
Full Viterbi Procedure

Input: scores $s(x, i, y, y')$, for all $i \in \{0, \ldots, n\}$, $y, y' \in \mathcal{L}$

Output: $\hat{y}$

1. Base case: $\bigvee_1(y) = s(x, 0, \bigcirc, y)$
2. For $i \in \{2, \ldots, n + 1\}$:
   - Solve for $\bigvee_i(*)$ and $bp_i(*)$.
     $$\bigvee_i(y) = \max_{y_{i-1} \in \mathcal{L}} s(x, i - 1, y_{i-1}, y) + \bigvee_{i-1}(y_{i-1}),$$
     $$bp_i(y) = \arg\max_{y_{i-1} \in \mathcal{L}} s(x, i - 1, y_{i-1}, y) + \bigvee_{i-1}(y_{i-1})$$
     (At $n + 1$ we’re only interested in $y = \bigcirc$.)
3. $\hat{y}_{n+1} \leftarrow \bigcirc$
4. For $i \in \{n, \ldots, 1\}$:
   - $\hat{y}_i \leftarrow bp_{i+1}(\hat{y}_{i+1})$
Viterbi Asymptotics

<table>
<thead>
<tr>
<th>labels in $\mathcal{L}$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$\ldots$</th>
<th>$x_n$</th>
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</thead>
<tbody>
<tr>
<td>$\ell_1$</td>
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<tr>
<td>$\ell_2$</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>$\vdots$</td>
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</tr>
<tr>
<td>$\ell_L$</td>
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</tbody>
</table>
Viterbi Asymptotics

Space: need to store $s$, and fill in the cells above.
Viterbi Asymptotics

<table>
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<th>labels in $\mathcal{L}$</th>
<th>$x_1$</th>
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<tr>
<td>$\ell_2$</td>
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</tr>
<tr>
<td>$\vdots$</td>
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<td></td>
</tr>
<tr>
<td>$\ell_L$</td>
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</tbody>
</table>

Space: need to store $s$, and fill in the cells above. $O(nL^2)$ for $s$ (in the most general case, often less), $O(nL)$ for cells
Viterbi Asymptotics

<table>
<thead>
<tr>
<th>labels in $\mathcal{L}$</th>
<th>$\ell_1$</th>
<th>$\ell_2$</th>
<th>\vdots</th>
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<td>$x_2$</td>
<td>\ldots</td>
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</tr>
</tbody>
</table>

Space: need to store $s$, and fill in the cells above. $O(nL^2)$ for $s$ (in the most general case, often less), $O(nL)$ for cells

Runtime: each cell requires an “argmax.”
Viterbi Asymptotics

<table>
<thead>
<tr>
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<th>$x_2$</th>
<th>$\ldots$</th>
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<tbody>
<tr>
<td>$\ell_1$</td>
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<tr>
<td>$\ell_2$</td>
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<tr>
<td>$\vdots$</td>
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<tr>
<td>$\ell_L$</td>
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</tbody>
</table>

Space: need to store $s$, and fill in the cells above. $O(nL^2)$ for $s$ (in the most general case, often less), $O(nL)$ for cells

Runtime: each cell requires an “argmax.” $O(nL^2)$
Why it Works

Viterbi exploits the distributivity property:

$$\max_{y_{1:n}} \sum_{i=0}^{n} s(x, i, y_i, y_{i+1}) = \max_{y_n} s(x, i, y_n, \bigcirc) + \max_{y_{1:n-1}} \sum_{i=0}^{n-1} s(x, i, y_i, y_{i+1})$$

$$= \max_{y_n} s(x, i, y_n, \bigcirc) + \max_{y_{n-1}} s(x, i, y_n - 1, y_n)$$

$$+ \max_{y_{1:n-2}} \sum_{i=0}^{n-2} s(x, i, y_i, y_{i+1})$$

Max plus max plus max plus max plus max plus ...
We haven’t said much about the function that scores candidate label pairs at different positions, $s(x, i, y, y')$.

This function is very important; two common choices are:

- Expert-designed, task-specific features $f(x, i, y, y')$ and weights $\theta$
- A neural network that encodes $x_i$ in context, $y_i$, and $y_{i+1}$ and gives back a goodness score

Either way, let $\theta$ denote the parameters of $s$. From now on, we’ll use $s(x, i, y, y'; \theta)$ and $\text{Score}(x, y; \theta)$ to emphasize that “$s$” is a function of parameters $\theta$ we need to estimate.
As we’ve done before, we start with the principle of maximum likelihood to estimate $\theta$:

$$\theta^* = \arg \max_{\theta \in \mathbb{R}^d} \prod_{i=1}^T p(Y = y_i \mid X = x_i; \theta)$$

$$= \arg \max_{\theta \in \mathbb{R}^d} \sum_{i=1}^T \log p(Y = y_i \mid X = x_i; \theta)$$

$$= \arg \min_{\theta \in \mathbb{R}^d} \sum_{i=1}^T -\log p(Y = y_i \mid X = x_i; \theta)$$

sometimes called “log loss” or “cross entropy”

Next, we’ll drill down into “$p(Y = y_i \mid X = x_i; \theta)$.”
Conditional Random Fields
Lafferty et al. (2001)

CRFs are a tremendously influential model that generalizes multinomial logistic regression to structured outputs like sequences.

\[ p_{\text{CRF}}(y \mid x; \theta) = \frac{\exp \text{Score}(x, y; \theta)}{Z(x; \theta)} \]

\[ Z(x; \theta) = \sum_{y' \in \mathcal{Y}(x)} \exp \text{Score}(x, y'; \theta) \]

\[ -\log p_{\text{CRF}}(y \mid x; \theta) = -\text{Score}(x, y; \theta) + \log Z(x; \theta) \]

So, our “CRF”:

- Uses Viterbi for decoding (our v. 4 sequence labeler)
- Trains parameters to maximize likelihood (like MLR and NNs)
Conditional Random Field
Lafferty et al. (2001)
Sequence-Level Log Loss

Here’s the maximum likelihood learning problem (equivalently, sequence-level log loss):

$$\theta^* = \arg\min_{\theta \in \mathbb{R}^d} \sum_{i=1}^{T} -\text{Score}(x_i, y_i; \theta) + \log Z(x_i; \theta)$$

If we can calculate and differentiate (w.r.t. $\theta$) the $\text{Score}$ and $Z$ functions, we can use SGD to learn.
Reflection

Given a training instance $\langle x_i, y_i \rangle$, what do you need to do to calculate $\text{Score}(x_i, y_i; \theta)$?
Calculating $Z(x; \theta)$

Good news! The algorithm that gives us $Z$ is *almost exactly like* the Viterbi algorithm.

Forward algorithm: sums the $\exp \text{Score}$ values for all label sequences, given $x$, in the same asymptotic time and space as Viterbi.

Let $\alpha_i(y)$ be the sum of all (exponentiated) scores of label prefixes of length $i$, ending in $y$. 
Some Algebra

Given the decomposition

\[ \text{Score}(x, y; \theta) = \sum_{i=0}^{n} s(x, i, y_i, y_{i+1}; \theta), \]

it holds that

\[ \exp \text{Score}(x, y; \theta) = \prod_{i=0}^{n} e^{s(x, i, y_i, y_{i+1}; \theta)}, \]

and therefore

\[ Z(x; \theta) = \sum_{y' \in \mathcal{Y}(x)} \prod_{i=0}^{n} e^{s(x, i, y'_i, y'_{i+1}; \theta)} \]
Forward Algorithm

Input: scores $s(x, i, y, y'; \theta)$, for all $i \in \{0, \ldots, n\}$, $y, y' \in \mathcal{L}$

Output: $Z(x; \theta)$

1. Base case: $\alpha_1(y) = e^{s(x, 0, \bigcirc, y; \theta)}$
2. For $i \in \{2, \ldots, n + 1\}$:
   ▶ Solve for $\alpha_i(\ast)$.
   \[
   \alpha_i(y) = \sum_{y_{i-1} \in \mathcal{L}} e^{s(x, i-1, y_{i-1}, y; \theta)} \times \alpha_{i-1}(y_{i-1})
   \]
   (At $n + 1$ we’re only interested in $y = \bigcirc$.)
3. Return $\alpha_{n+1}(\bigcirc)$, which is equal to $Z(x; \theta)$. 
Intuitions about the Forward Algorithm

Just as Viterbi changes “scary max over big sum” to “max plus max plus max plus . . . ,”
the Forward algorithm changes “scary sum over big product” to
“plus times plus times plus times plus times . . . .”

If you organize the operations in the other direction, you get the Backward algorithm.

You can differentiate $Z$ with respect to $s$, because it’s all just $\exp$, addition, and multiplication. If you mechanically derive the partial derivatives, you will rediscover the Backward algorithm.
Computation Graph View of CRF

\[
\begin{align*}
\text{loss}(\theta) &= -\log p_{\text{CRF}}(y | x; \theta) - \text{Viterbi loss}(\theta) + \log Z(x; \theta) \\
&= \text{Forward} - \text{Score} - \text{Viterbi}\end{align*}
\]
Earlier in the lecture, I promised that learning would have some guarantees. Consider:

- The runtime and space requirements for calculating the loss and gradient, as a function of the data.
- The conditions under which we can confidently expect convergence to a global optimum of the likelihood if we use SGD.
An Alternative: Structured Perceptron

Recall that CRF = v. 4 + sequence-level log loss.

Perceptron loss (Collins, 2002):

$$\theta^* = \arg\min_{\theta \in \mathbb{R}^d} \sum_{i=1}^{T} -\text{Score}(x_i, y_i; \theta) + \max_y \text{Score}(x_i, y; \theta)$$

The structured perceptron = v. 4 + perceptron loss.
Regularization

Just as in classification with linear and non-linear models, you’ll want to take steps to avoid overfitting.

The same tools (e.g., $\ell_2$ and $\ell_1$ penalties for linear model weights, and dropout for neural networks) can be used here.
V. 2–4 are weighted finite-state machines (think of labels as states).

The models we saw today are all “first order” sequence models in the sense that each $y_i$ only interacts with one immediate neighbor through $s$.

- **Second-order**: $\text{Score}(x, y) = \sum_{i=0}^{n} s(x, i, y_i, y_{i+1}, y_{i+2})$
- **$m$th-order**: $\text{Score}(x, y) = \sum_{i=0}^{n} s(x, i, y_{i:i+m})$

Viterbi for $m$th order has $O(nL^{m+1})$ runtime.


References II


